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OPTIMUM REDUNDANT CONFIGURATIONS OF INERTIAL SENSORS

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13. ABSTRACT

Within the present state of the art inertial instruments can achieve high reliability for long periods of time only by redundancy. By suitable geometric configurations it is possible to extract the maximum amount of reliability and accuracy from a given munber of redundant single-degree-of-freedom gyros or accelerometers.

This paper gives a general derivation of the optimum matrix which can be applied to the outputs of any combination of 3 or more instruments to obtain 3 orthogonal vector components based on their geometric configuration and error characteristics. The result is a special case of Kalman filtering theory. Certain combinations of 4 or more instruments have the capability of detecting an instrument malfunction, those of 5 or more have the additional capability of isolating that malfunction to a particular instrument.

This paper gives particularly attractive configurations of 4, 5 and 6 instruments. These combinztions are capable of functioning with any 3 instruments, of detecting a malfunction with any 4, and of isolating a malfunction with ahy 5. In contrast, arrangements with redundant instruments whose input axes are parallel to only 3 orthogonal axes require 6 instruments to detect and 9 to isolate a malfunction. Also given are the matrices to be applied to the instrument outputs to detect and isolate more malfunctions. The instrument configurations shown not only minimize system error when all instruments are operable, but when all but 3 have malfunctioned as well.

The reliability and accuracy of these redundant configurations are compared in this paper to a conventional unit of 3 orthogonal instruments. They are also compared with

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redundant sets of such units. Consideration is given to cases where an external reference is or is not available to detect and isolate malfunctions. Although the improvement in accuracy is minor, the improvement in reliability is major. As an example, if no external reference is available to isolate malfunctions, then 5 instruments arranged in the given configuration are as reliable as a triple modular redundant combination of conventional units of 3 instruments (9 instruments in all).

Although consideration of the use of redundant orthogonal instrument triads is common, the author knows of no proposal that more than 3 instruments be combined in a single package with axes skewed in such a manner that each makes an equal contribution to the total system performance. The principles contained in the paper have an immediate applicabtion to strapdown attitude reference packages for space vehicles. However, they are also applicable to gimballed stable platforms and to accelerometer packages.

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OPTIMUM REDUNDANT CONFIGURATIONS OF INERTIAL SENSORS

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ABSTRACT

Within the present state-of-the-art inertial instruments can achieve high reliability for long periods of time only by redundancy. By suitable geometric configurations it is possible to extract the maximum amount of reliability and accuracy from a given number of redundant single-degree-of-freedom gyros or accelerometers.

This paper gives a general derivation of the optimum matrix which can be applied to the outputs of any combination of 3 or more instruments to obtain 3 orthogonal vector components based on their geometric configuration and error characteristics. The result is a special case of Kalman filtering theory. Certain combinations of 4 or more instruments have the capability of detecting an instrument malfunction, those of 5 or more have the additional capability of isolating that malfunction to a particular instrument.

This paper gives particularly attractive configurations of 4, 5, and 6 instruments. These combinations are capable of functioning with any 3 instruments, of detecting a malfunction with any 4, and of isolating a malfunction with any 5. In contrast, arrangements with redundant instruments whose input axes are parallel to only 3 orthogonal axes require 6 instruments to detect and 9 to isolate a malfunction. Also given are the matrices to be applied to the instrument outputs to detect and isolate malfunctions and to operate the system after one or more malfunctions. The instrument configurations shown not only minimize system error when all instruments are operable, but when all but 3 have malfunctioned as well.

The reliability and accuracy of these redundant configurations are compared in this paper to a conventional unit of 3 orthogonal instruments. They are also compared with redundant sets of such units. Consideration is given to cases where an external reference is or is not available to detect and isolate malfunctions. Although the improvement in accuracy is minor, the improvement in reliability is major. As an example, if no external reference is available to isolate malfunctions, then 5 instruments arranged in the given configuration are as reliable as a triple modular redundant combination of conventional units of 3 instruments (9 instruments in all).

Although consideration of the use of redundant orthogonal instrument triads is common, the author knows of no proposal that more than 3 instruments be combined in a single package with axes skewed in such a manner that each makes an equal contribution to the total system performance. The principles contained in the paper have an immediate application to strapdown attitude reference packages for space vehicles. However they are also applicable to gimballed stable platforms and to accelerometer packages.

I. Introduction

Redundancy is a technique commonly used to insure high reliability over a long operating lifetime. In the field of inertial navigation, the most familiar example is the SINS which uses three complete inertial platforms. Some missile systems have a redundant gyroscope mounted on a single platform, but use it primarily for preflight azimuth alignment. The advanced SINS uses a fourth or "monitor" gyro to continuously calibrate the drift rate of the two horizontal gyros. It still relies on multiple platforms for reliability.

Most present analyses indicate that two redundant attitude reference units are required for a space navigation system having high probability of successful operation for more than a few days. This paper will detail a more efficient method of obtaining reliability through redundant gyroscopes in a single strapdown attitude reference unit. The same principles apply to gimballed gyros and to accelerometers.

II. Multi-gyro Analysis

A general block diagram of a strapdown attitude reference unit is shown in Figure 1.

The symbols used in Figure 1 have the following significance:

ω = vehicle angular rate vector

ω' = indicated angular rate vector

ϵ = gyro error torque vector

σ = gyro output signal vector

γ = indicated gyro error torque vector

A = matrix relating rotations about vehicle axes to those about gyro axes

B = matrix relating gyro output rates to indicated vehicle rates

C = matrix relating gyro output rates to indicated gyro error torques

h = gyro angular momentum

G(s) = gyro dynamic gain

K(s) = feedback gain

The general matrix equations which should hold for any configuration are:

$$BA = I \tag{1}$$

$$CA = 0 \tag{2}$$

The loop in Figure 1 can be reduced to:

$$h\omega' = B\frac{GK}{1 + GK}(Ah\omega + \epsilon) \approx h\omega + B\epsilon \tag{3}$$

$$\gamma = C\frac{GK}{1 + GK}(Ah\omega + \epsilon) \approx C\epsilon \tag{4}$$

The approximations shown above are based on system design which makes GK >>1 in the frequency range of interest.

For a conventional gyro package with 3 orthogonal input axes, the matrices are:

$$A = B = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

The matrix A relating rotations about the vehicle axes to those about the gyro axes is fixed by the physical arrangement of the gyro package. The matrix B relating gyro output rates to indicated vehicle rates should be chosen to minimize system error due to gyro drifts and malfunctions.

The mean square error is

$$E^2 = <\tilde{\epsilon}\,\tilde{B}\,B\epsilon> = \mathrm{tr}\left[\tilde{B}\,B<\epsilon\tilde{\epsilon}>\right] \tag{6}$$

where the brackets indicate an ensemble average, the tilde (~), the matrix transpose, and tr, the trace or sum of the diagonal elements of this matrix.

Minimizing E^2 subject to the constraint that BA = I gives the following matrix equation

$$B<\epsilon\tilde{\epsilon}> - \Lambda\tilde{A} = 0 \tag{7}$$

where Λ is a Lagrangian matrix multiplier. The solution of the equations is identical to the Kalman theory results.

$$\Lambda = \left[\tilde{A}<\epsilon\tilde{\epsilon}>^{-1}A\right]^{-1} \tag{8}$$

$$B = \left[\tilde{A}<\epsilon\tilde{\epsilon}>^{-1}A\right]^{-1}\tilde{A}<\epsilon\tilde{\epsilon}>^{-1} \tag{9}$$

Substituting Equation (9) in Equation (6) gives

$$E^2 = \mathrm{tr}\Lambda \tag{10}$$

as the minimum error possible with a physical arrangement represented by the matrix A and gyro errors represented by the vector ϵ.

If the gyro errors are independent and have equal variances (e^2) from zero means, then

$$<\epsilon\tilde{\epsilon}> = e^2 I \tag{11}$$

$$B = \left[\tilde{A}\,A\right]^{-1}\tilde{A} \tag{12}$$

$$E^2 = e^2\mathrm{tr}\left[\tilde{A}\,A\right]^{-1} \tag{13}$$

The value of E^2 may be further minimized by a physical arrangement of n gyros which makes the columns of the A matrix orthogonal. Since each row of A represents a set of direction cosines, it follows that

$$\tilde{A}\,A = \frac{n}{3}I \tag{14}$$

$$B_o = \frac{3}{n}\tilde{A} \tag{15}$$

$$E_o^2 = \frac{9}{n}e^2 \tag{16}$$

For $n > 3$, the requirement that the columns of A be orthogonal does not uniquely define a configuration for a multi-gyro package. It is desired to minimize system errors after all but 3 gyros have malfunctioned as well as when all n are operating. The author knows of no general method of obtaining an optimum configuration. However, symmetry has been used to discover configurations which appear to be optimum for the cases of 4, 5, and 6 gyros.

The choice of a matrix B to use after one or more gyros have malfunctioned is based on Equation (9) except that the covariance matrix $<\epsilon\tilde{\epsilon}>$ is modified by placing infinitely large values in the terms corresponding to the malfunctioned gyros. Identical results can be obtained by modifying the matrix A by replacing the direction cosines of the malfunctioned gyros by zeros.

The matrix C relating gyro output rates to indicated gyro drift errors should be chosen to extricate the maximum amount of information on gyro drifts from the redundant information available. The mean square error in the estimated gyro error is

$$F^2 = <(\tilde{\epsilon} - \tilde{\gamma})(\epsilon - \gamma)> = \mathrm{tr}\left[(I - \tilde{C})(I - C)<\epsilon\tilde{\epsilon}>\right] \quad (17)$$

Minimizing F^2 subject to the constraint $CA = 0$ gives the following matrix equation

$$(I - C) <\epsilon\epsilon> - \Lambda'\tilde{\Lambda} = 0 \quad (18)$$

where Λ' is another Lagrangian matrix multiplier. The solution of the equation is similar to that for the matrix B.

$$\Lambda' = A\left[\tilde{A} <\epsilon\tilde{\epsilon}>^{-1} A\right]^{-1} \quad (19)$$

$$C = I - AB \quad (20)$$

The matrix C has the additional properties

$$BC = CA = 0 \quad (21)$$

$$C^2 = C \quad (22)$$

With 4 or more gyros in operation, a malfunction will cause one or more components of γ to exceed a pre-set threshold. With 5 or more gyros it is possible to isolate the malfunction to a particular gyro which corresponds to the maximum component of γ.

III. Other Applications

The above analysis for strapdown gyros is directly applicable to either strapdown or gimballed accelerometer packages as well. The appropriate meaning for the symbols in Figure 1 for both pendulous and test-mass accelerometers are given in Table I below.

		ACCELEROMETER	
SYMBOL	GYROSCOPE	TORQUE BALANCE	FORCE BALANCE
ω	angular rate	acceleration	acceleration
ϵ	error torque	error torque	error force
σ	output torque	output torque	output force
γ	indicated error torque	indicated error torque	indicated error force
h	angular momentum	pendulosity	test mass
G(s)	gyro gain	accelerometer gain	accelerometer gain

TABLE I.

By partially caging the gyros, redundancy can also be used in a gimballed stable platform as shown in Figure 2.

The additional symbols used in Figure 2 have the following significance

$\delta\omega$ = platform drift rate vector

$K_1(s)$= platform feedback gain

$K_2(s)$= gyro torque feedback gain

The loop in Figure 2 can be reduced to:

$$h\delta\omega = \frac{h}{1 + K_1Gh}(\omega - K_1GB\epsilon) \approx -B\epsilon \quad (23)$$

$$\gamma = K_2GC(1 - \frac{K_2G}{1 + K_2G}C)\epsilon = \frac{K_2G}{1 + K_2G}C\epsilon \approx C\epsilon \quad (24)$$

where as before the approximations are based on system design which makes $GK >> 1$ in the frequency range of interest. As indicated, the effects of redundancy on system mechanization are different for gimballed systems, but much of the analysis and results are identical to that for strapdown systems.

IV. Four-Gyro Configuration

The most symmetric configuration of four gyro input axes is the normals to the faces of

a regular tetrahedron. The angle between any two input axes would then be 109° 28' (arctan - 2 $\sqrt{2}$). The directions of the gyro output axes are to a certain extent arbitrary, but are often chosen either to minimize acceleration sensitive errors or simply for packaging reasons. The direction cosines of the gyro axes for one such configuration (Figure 3) are given in Table II below.

	Input Axis			Output Axis			Spin Axis		
Gyro No.	X	Y	Z	X	Y	Z	X	Y	Z
1	1	0	0	0	0	1	0	-1	0
2	-.333	.943	0	0	0	-1	-.943	-.333	0
3	-.333	-.471	.816	.816	-.816	0	.471	.667	.577
4	-.333	-.471	-.816	-.816	.816	0	.471	.667	-.577

TABLE II.

V. Four-Gyro Analysis

For the four-gyro configuration the matrix relating rotations about the vehicle axes to those about gyro axes is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -.333 & .943 & 0 \\ -.333 & -.471 & .816 \\ -.333 & -.471 & -.816 \end{bmatrix} \tag{25}$$

The matrix which relates gyro output rates to indicated vehicle rates when all gyros are operating properly is given by

$$B_o = \begin{bmatrix} .75 & -.25 & -.25 & -.25 \\ 0 & .707 & -.354 & -.354 \\ 0 & 0 & .612 & -.612 \end{bmatrix} \tag{26}$$

A check can be made on the gyro cluster by combining the output of the 4 gyros in such a way that the physical angular rates cancel and the check vector depends only on the gyro torques. The appropriate check matrix is:

$$C_o = \begin{bmatrix} .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \end{bmatrix} \tag{27}$$

Assuming that gyro No. 1 malfunctions, the error torque vector $\gamma = C\sigma$ will exceed a threshold. With external instrumentation such as a star tracker to localize the malfunction, the bad gyro can be switched out of the system by changing matrix B as shown below.

$$B_1 = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & .707 & -.354 & -.354 \\ 0 & 0 & .612 & -.612 \end{bmatrix} \tag{28}$$

With only 3 gyros operating, an internal check no longer exists.

VI. Five-Gyro Configuration

The most effective configuration would have no 3 gyros in the same plane so that the package is still useful after 2 gyro failures. For reasons of symmetry, a configuration in which all 5 input axes make equal angles with a given axis is chosen. The specific angle which the input axes make with the central axis is chosen so as to minimize the effect of individual gyro errors on total system errors for the worst case when 2 gyros are inoperative.

The angle which does optimize the configuration is 54° 44' (arctan$\sqrt{2}$) with the axis of symmetry. The resulting angle between input axes is 57° 22'. The direction cosines of the gyro axes for one such configuration (Figure 4) are given in Table III below.

	Input Axis			Output Axis			Spin Axis		
Gyro No.	X	Y	Z	X	Y	Z	X	Y	Z
1	.577	.816	0	0	0	1	.816	-.577	0
2	.577	.252	.777	.400	-.916	0	.712	.311	-.630
3	.577	-.661	.480	.753	.658	0	-.316	.361	.877
4	.577	-.661	-.480	.753	.658	0	.316	-.361	.877
5	.577	.252	-.777	.400	.916	0	-.712	-.311	-.630

TABLE III.

VII. Five-Gyro Analysis

For the five -gyro configuration the matrix relating rotations about the vehicle axes to those about gyro axes is:

$$
A = \begin{bmatrix} .577 & .816 & 0 \\ .577 & .252 & .777 \\ .577 & -.661 & .480 \\ .577 & -.661 & -.480 \\ .577 & .252 & -.777 \end{bmatrix} \tag{29}
$$

The matrices which relate gyro output rates to indicated vehicle rates and error torques when all gyros are operating properly are:

$$
B_0 = \begin{bmatrix} .346 & .346 & .346 & .346 & .346 \\ .490 & .151 & -.396 & -.396 & .151 \\ 0 & .466 & .288 & -.288 & -.466 \end{bmatrix} \tag{30}
$$

$$
C_0 = \begin{bmatrix} .4 & -.324 & .124 & .124 & -.324 \\ -.324 & \text{[illegible]} & -.324 & .124 & \text{[illegible]} \\ .124 & -.324 & \text{[illegible]} & -.324 & .124 \\ .124 & .124 & -.324 & .4 & -.324 \\ -.324 & .124 & .124 & -.324 & .4 \end{bmatrix} \tag{31}
$$

Assuming that gyro No. 1 malfunctions, one or more of the elements of the check vector y will exceed a predetermined threshold. The first element will be the largest and therefore the corresponding gyro should be switched out of the system by changing matrices B and C as shown below.

$$
B_1 = \begin{bmatrix} 0 & .627 & .239 & .239 & .627 \\ 0 & .548 & -.548 & -.548 & .548 \\ 0 & .466 & .288 & -.288 & -.466 \end{bmatrix} \tag{32}
$$

$$
C_1 = \begin{bmatrix} \text{[illegible]} & -.809 & .309 & .309 & -.809 \\ 0 & .138 & -.224 & .224 & -.138 \\ \text{[illegible]} & -.224 & .362 & -.362 & .224 \\ 0 & .224 & -.362 & .362 & -.224 \\ 0 & -.138 & .224 & -.224 & .138 \end{bmatrix} \tag{33}
$$

It is still possible to detect a malfunction. However, it can no longer be localized to a single gyroscope without the aid of external instrumentation such as a star tracker.

If 2 adjacent gyros such as Nos. 3 and 4 malfunction, the matrix which eliminates their effect is

$$
B_{34} = \begin{bmatrix} -.775 & 1.253 & 0 & 0 & 1.253 \\ 1.772 & -.886 & 0 & 0 & -.886 \\ 0 & .644 & 0 & 0 & -.644 \end{bmatrix} \tag{34}
$$

If 2 non-adjacent gyros such as No. 2 and 5 malfunction, the matrix which eliminates their effect is

$$
B_{25} = \begin{bmatrix} .775 & 0 & .479 & .479 & 0 \\ .677 & 0 & -.339 & -.339 & 0 \\ 0 & 0 & 1.042 & -1.042 & 0 \end{bmatrix} \tag{35}
$$

VIII. Six-Gyro Configuration

Considerations similar to those discussed for the four- and five-gyro attitude reference units govern the choice of a configuration for 6 gyroscopes. The resulting configuration is one in which all input axes make angles of either 63° 26' (arctan 2) or 116° 34' with each other. Geometrically the input axes are perpendicular to the faces of a regular dodecahedron. The direction cosines of the gyro axes for one such configuration (Figure 3) are given in Table IV below.

Gyro No.	Input Axis X	Input Axis Y	Input Axis Z	Output Axis X	Output Axis Y	Output Axis Z	Spin Axis X	Spin Axis Y	Spin Axis Z
1	.795	.607	0	0	0	1	.607	-.795	0
2	.795	-.304	.526	0	-.866	-.5	.607	.397	-.688
3	.795	-.304	-.526	0	.866	-.5	.607	.397	.688
4	-.188	.982	0	0	0	1	-.982	-.188	0
5	-.188	-.491	.851	0	-.866	-.5	-.982	.094	-.162
6	-.188	-.491	-.851	0	.866	-.5	-.982	.094	.162

TABLE IV.

IX. Six-Gyro Analysis

For the six-gyro configuration the matrix relating rotations about the vehicle axes to those about gyro axes is:

$$A = \begin{bmatrix} .795 & .607 & 0 \\ .795 & -.304 & .526 \\ .795 & -.304 & -.526 \\ -.188 & .982 & 0 \\ -.188 & -.491 & .851 \\ -.188 & -.491 & -.851 \end{bmatrix} \tag{36}$$

The matrices which relate gyro output rates to indicated vehicle rates and error torques when all gyros are operating properly are:

$$B_o = \begin{bmatrix} .397 & .397 & .397 & -.094 & -.094 & -.094 \\ .304 & -.152 & -.152 & .491 & -.246 & -.246 \\ 0 & .263 & -.263 & 0 & .425 & -.425 \end{bmatrix} \tag{37}$$

$$C_o = \begin{bmatrix} .5 & -.224 & -.224 & -.224 & .224 & .224 \\ -.244 & .5 & -.224 & .224 & -.224 & .224 \\ -.224 & -.224 & .5 & .224 & .224 & -.224 \\ -.224 & .224 & .224 & .5 & -.224 & -.224 \\ .224 & -.224 & .224 & -.224 & .5 & -.224 \\ .224 & .224 & -.224 & -.224 & -.224 & .5 \end{bmatrix} \tag{38}$$

Assuming the gyro No. 1 malfunctions, one or more of the elements of the check vector $\gamma = C\sigma$ will exceed a predetermined threshold. The first element will be the largest and therefore gyro No. 1 should be switched out of the system by changing matrices B and C as shown below.

$$B_1 = \begin{bmatrix} 0 & .575 & .575 & .083 & -.272 & -.272 \\ 0 & -.016 & -.016 & .627 & -.381 & -.381 \\ 0 & .263 & -.263 & 0 & -.425 & -.425 \end{bmatrix} \tag{39}$$

$$C_1 = \begin{bmatrix} 1 & -.447 & -.447 & -.447 & .447 & .447 \\ 0 & .4 & -.324 & .124 & -.124 & .324 \\ 0 & -.324 & .4 & .124 & .324 & -.124 \\ 0 & .124 & .124 & .4 & .324 & .324 \\ 0 & -.124 & .324 & .324 & .4 & .124 \\ 0 & .324 & -.124 & .324 & .124 & .4 \end{bmatrix} \tag{40}$$

It is still possible to isolate the second gyro malfunction by a method similar to that used to isolate the first.

If 2 gyros such as No. 5 and 6 have failed, the matrix which eliminates their effect is

$$B_{56} = \begin{bmatrix} .455 & .397 & .397 & -.036 & 0 & 0 \\ .455 & -.152 & -.152 & .643 & 0 & 0 \\ 0 & .951 & -.951 & 0 & 0 & 0 \end{bmatrix} \tag{41}$$

The check matrix is now

$$C_{56} = \begin{bmatrix} .362 & -.224 & -.224 & -.362 & 0 & 0 \\ -.224 & .138 & .138 & .224 & 0 & 0 \\ -.224 & .138 & .138 & .224 & 0 & 0 \\ -.362 & .224 & .224 & .362 & 0 & 0 \\ .309 & -.809 & .809 & .309 & 1 & 0 \\ .309 & .809 & -.809 & .309 & 0 & 1 \end{bmatrix} \tag{42}$$

The check procedure allows malfunction detection but not isolation when only 4 gyros are operative.

If 3 adjacent gyros such as No. 1, 2, and 3 malfunction and external methods are available to isolate the last malfunction, then the system can still be operated with the matrix

$$B_{123} = \begin{bmatrix} 0 & 0 & 0 & -1.777 & -1.777 & -1.777 \\ 0 & 0 & 0 & .679 & -.339 & -.339 \\ 0 & 0 & 0 & 0 & .588 & -.588 \end{bmatrix} \tag{43}$$

Similarly if 3 non-adjacent gyros such as No. 4, 5, and 6 malfunction, the appropriate matrix is

$$B_{456} = \begin{bmatrix} .419 & .419 & .419 & 0 & 0 & 0 \\ 1.098 & -.549 & -.549 & 0 & 0 & 0 \\ 0 & .951 & -.951 & 0 & 0 & 0 \end{bmatrix} \tag{44}$$

X. Reliability Analysis

For a configuration of n + m equal elements, only m of which are required for operation, the probability of successful operation is

$$P_{mn} = \sum_{k=0}^{n} \frac{(m+n)!}{k!\,(m+n-k)!} P_1^{m+n-k} (1 - P_1)^k \quad (45)$$

where P_1 is the probability of success of a single element. If the failure rate equals a constant λ, then

$$P_1 = \exp(-\lambda t) \quad (46)$$

For a multi-gyro attitude reference unit in which external sensors can resolve malfunctions to a particular gyro, successful operation of the unit is possible as long as 3 or more gyros are operating. In this case $m = 3$. If no external aid is available, the unit itself can resolve malfunctions as long as 4 or more gyros are operational. In this case $m = 4$.

For comparison, the reliability of N conventional 3-gyro attitude reference units in which only one gyro on each axis is operating at a time is given by

$$P_{3N} = \left[\sum_{k=0}^{N-1} \frac{(\lambda t)^k}{k!}\right]^3 \exp(-3\,\lambda t) \quad (47)$$

The above equation implies that external methods are available to detect and isolate a malfunction. A method which does not require external aid is triple modular redundancy. For a TMR configuration 3 gyros on each orthogonal axis are required. All 9 gyros operate concurrently and malfunctions are detected by comparing the outputs of each set of 3. When the output of a gyro varies by more than a preset tolerance from that of the other 2 corresponding gyros, its output is ignored. The failure of 2 gyros on a single axis is required to cause the failure of the total attitude reference unit. For TMR the reliability is given by

$$P_{TMR} = (P_{21})^3 = \left[3 \exp(-2\lambda t) - 2 \exp(-3\lambda)\right]^3 \quad (48)$$

Assuming a failure rate of $\lambda = 55.6 \times 10^{-6}$ failure/hour or an MTBF of 750 days, the times at which a reliability of 0.99 and 0.95 are reached are given in Table V below.

No. of Units	Total No. of Gyros	P = 0.99		P = 0.95	
		With Aid	w/oAid	With Aid	w/oAid
-	1	-	7.5 days	-	38 days
1	3	-	25	-	13
1	4	32 days	1.9	77 days	9.6
1	5	84	25	158	60
1	6	142	66	238	124
2	6	63	-	147	-
3	9	219	26	399	60

TABLE V.

In Table V the No. of Units refers to the number of separate attitude reference units. In the case of 2 or 3 each unit is a conventional three gyro package whose input axes are orthogonal to each other and parallel to the corresponding axes of the other units. With and without aid refers to whether or not an external device such as a star tracker is available to localize a malfunction to a particular gyro. With non-redundant systems, the external device is of no use since no spare is available to replace a malfunctioned gyro. In the cases of 2 or 3 conventional units with external aid to localize malfunctions, each gyro is considered to be individually replaceable and only one gyro on each axis is operable at a time. In the case of 3 units without aid the gyros are used in a TMR mode.

XI. Error Analysis

Based on Equation (6) for multi-gyro attitude reference units, the effective drift rate is given in Table VI below normalized so that the drift of a conventional three-gyro unit is unity.

Number of Gyros		Error	
Total	Operating	Mean Square	RMS
3	3	1	1
4	4	0.75	0.87
	3	1.5	1.22
5	5	0.6	0.78
	4	0.9	0.95
	3	2.2	1.48
6	6	0.5	0.71
	5	0.67	0.82
	4	1	1
	3	2.5	1.58

TABLE VI.

For the cases of five- and six-gyro units with only 3 gyros operating, there is a variation in total drift rate depending upon whether the active gyro input axes are adjacent or non-adjacent. The value given for the mean square error is an average of the two. Both cases are equally likely.

It may be noted that in no case does the error of a multi-gyro unit exceed that of a conventional three-gyro unit when 4 or more gyros are operating.

XII. Adaptive System Analysis

Since the value of the B matrix is based on estimated relative instrument errors and the check vector γ is itself an estimate of these errors, the development of an adaptive system mechanization is suggested. Unfortunately such an adaptive system in which the weighting factors are dependent on system outputs is non-linear and not amenable to the known techniques of linear system analysis.

Qualitatively it is apparent that the efficiency of a non-linear adaptive system depends on the degree of redundancy available. A system of 4 sensors provides no information to resolve individual errors; a system of 5 only enough to resolve errors so large as to be termed malfunctions. A system of 6 is probably the minimum degree of redundancy required to use an adaptive system.

A likely algorithm for an adaptive system would be to periodically recompute the matrix B based on the equations below.

$$B = \left[\tilde{A}Q^{-1}A\right]^{-1} \tilde{A}Q^{-1} \tag{49}$$

$$C = I - AB \tag{50}$$

where Q is a diagonal matrix with elements equal to $e^2(i) + \gamma^2(i)$. The matrix Q replaces the covariance matrix $<\epsilon\tilde{\epsilon}>$. The value $e(i)$ represents the a priori estimate of instrument error while $\gamma(i)$ is the current estimate based on Equation (4).

With insufficient redundancy or particular combinations of errors the above procedure may lead to instability, in other cases to a stable but non-optimum result. As a result of the non-linearity of the problem, drawing quantitative stability boundaries and determining the behavior for generalized error inputs requires either a great deal of numerical calculations or an advance in the analytical state-of-the-art. This problem is a variant of the third unsolved problem outlined by R. C. K. Lee in his book Optimal Estimation, Identification, and Control (MIT Press, Cambridge, 1964, p. 139).

XIII. Conclusions

1. Using redundant gyros in an optimal configuration in a single attitude-reference unit is a more efficient method of increasing reliability than redundant conventional three-gyro units.

2. Four or more gyros are required to detect an individual instrument failure without an external reference.

3. Five or more gyros are required to localize a malfunction to a particular instrument so that it may be switched out of the system and prevent system failure.

4. Six or more gyros are probably required to implement an adaptive system which would compensate for large instrument errors as well as complete failures.

5. The same principles may be used for any set of single-degree-of-freedom sensors used to detect a vector quantity.

6. Although the primary purpose is to increase reliability, redundancy also reduces total system error.

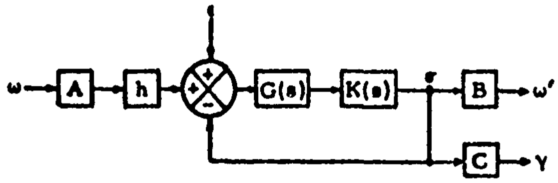

FIGURE 1. Block Diagram of Strapdown ARU

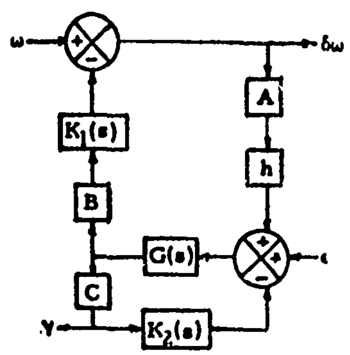

FIGURE 2. Block Diagram of Stable Platform

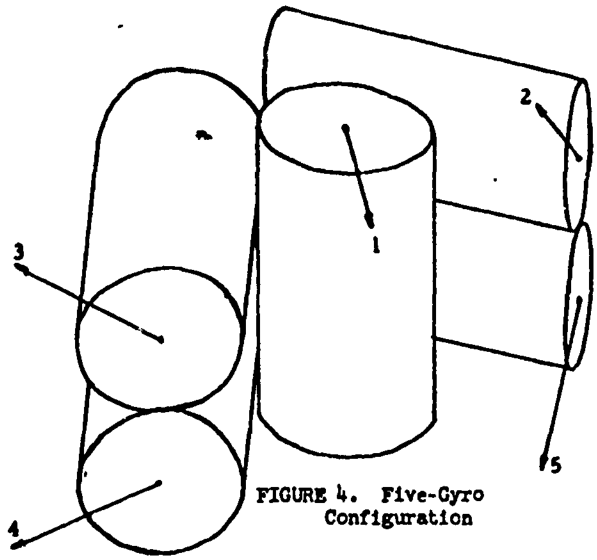

FIGURE 4. Five-Gyro Configuration

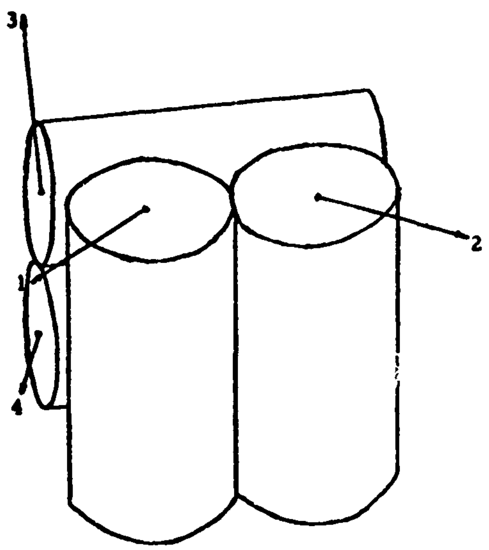

FIGURE 3. Four-Gyro Configuration

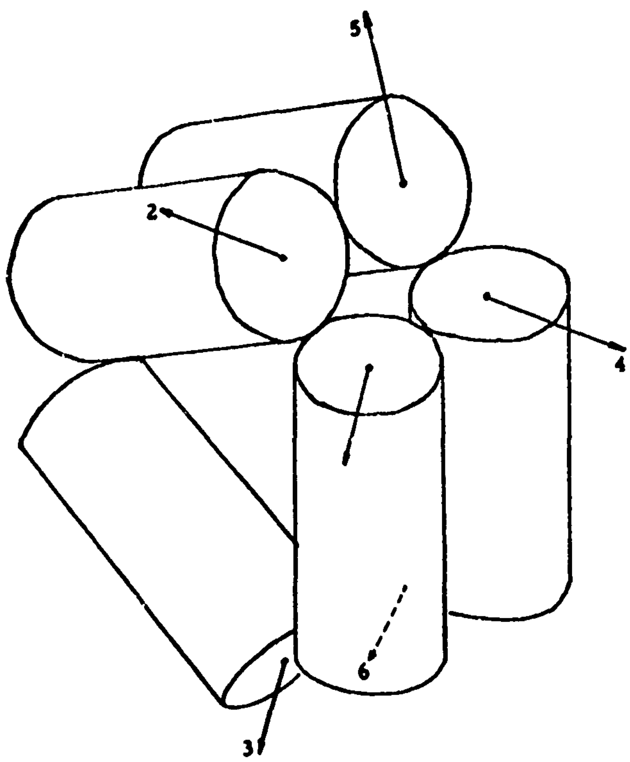

FIGURE 5. Six-Gyro Configuration